Algebra 2/Pre-Calculus

Name__________________

The Binomial Theorem (Day 4, Pascal's Triangle)

The goal of this handout is continue our exploration of Pascal's Triangle and to develop the binomial theorem.

Reminder of the connection to the Combination Numbers

Pascal's triangle is closely related to the combination numbers. Here's a reminder of how the combination numbers work:

$$\binom{n}{r} = {}_nC_r = \frac{n!}{(n-r)!r!}$$

The combination numbers are used when we are selecting a group with no repeats where order doesn't matter. For example, if I wanted to choose a group of three students from a class of 25, there are $\binom{25}{3} = 2300$ possible groups.

1. There are 10 members on the Portland City Council. The council members need to choose (from among themselves) a 3-person subcommittee.

a. How many different possibilities are there for the members of a 3-person subcommittee?

b. The Mayor of Portland is one of the 10 members of the City Council. How many different possibilities are there for the members of a 3-person subcommittee,
if the Mayor **must be included** as one of the subcommittee members?
Hint: The subcommittee will consist of the Mayor plus 2 of the other 9.

c. The Mayor of Portland is one of the 10 members of the City Council. How many different possibilities are there for the members of a 3-person subcommittee,
if the Mayor **must <u>not</u> be included** as one of the subcommittee members?

d. What combination number addition relationship is illustrated by parts **a**, **b,** and **c**?

2. In the last problem, you should have found that the answer to part **a** was ${}_{10}C_3 = 120$, the answer to part **b** was ${}_9C_2 = 36$ and the answer to part **c** was ${}_9C_3 = 84$. Note that ${}_9C_2 + {}_9C_3 = {}_{10}C_3$. Explain why this makes sense. ***Hint:*** The total number of 3 person committees can be thought of as two groups: all of the committees that include the mayor and all of the committees that do not include the mayor.

Explanation for Problem 2 We need to choose a committee of 3 from a group of 10 people, so there are ${}_{10}C_3 = 120$ total ways to do this. We can break these up into two groups: committees that include the mayor and committees that do not. If the mayor is not on the committee, we need to choose 3 people from the 9 remaining committee members. There are ${}_9C_3 = 84$ ways to do this. If the mayor is on the committee, we only need to choose 2 people from the remaining 9 people. There are ${}_9C_2 = 36$ ways to do this. Hence, the total number of ways to choose the committee is ${}_9C_2 + {}_9C_3 = {}_{10}C_3 = 120$.

3. The LHS Math department consists of 20 regular teachers and a department head (21 teachers total). The department head must choose 5 teachers to teach the Math 3 course. (The department head may or may not actually teach the course himself.) Explain what each of the following combination numbers means in the context of this situation.

a. ${}_{21}C_5$ **b.** ${}_{20}C_5$ **c.** ${}_{20}C_4$

d. Use the context of this problem situation to explain why ${}_{21}C_5 = {}_{20}C_5 + {}_{20}C_4$.

4. In this problem, we will generalize the result we developed in the last few problems. Carefully fill in the question marks to complete the following statement:

${}_{m}C_n = {}_{?}C_? + {}_{?}C_?$.

5. Find the values of *m* and *n* that satisfy each question. Remember, $\binom{a}{b}$ and ${}_aC_b$ mean the same thing.

a. $\binom{9}{4} = \binom{8}{4} + \binom{8}{m}$

b. $\binom{3}{2} = \binom{2}{2} + \binom{2}{m}$

c. $\binom{14}{8} = \binom{13}{8} + \binom{13}{m}$

d. $\binom{100}{15} = \binom{m}{15} + \binom{m}{n}$

6. Remember that Pascal's Triangle is made up of the combination numbers. We know that ${}_4C_1 + {}_4C_2 = {}_5C_2$. Explain how this is related to Pascal's Triangle.

$$
\begin{array}{c}
{}_0C_0 \\
{}_1C_0 \quad {}_1C_1 \\
{}_2C_0 \quad {}_2C_1 \quad {}_2C_2 \\
{}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 \\
{}_4C_0 \quad \mathbf{{}_4C_1} \quad \mathbf{{}_4C_2} \quad {}_4C_3 \quad {}_4C_4 \\
{}_5C_0 \quad {}_5C_1 \quad \mathbf{{}_5C_2} \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5 \\
{}_6C_0 \quad {}_6C_1 \quad {}_6C_2 \quad {}_6C_3 \quad {}_6C_4 \quad {}_6C_5 \quad {}_6C_6
\end{array}
\quad = \quad
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad \mathbf{4} \quad \mathbf{6} \quad 4 \quad 1 \\
1 \quad 5 \quad \mathbf{10} \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
$$

7. Find each of the following by multiplying. Simplify your answers by combining any like terms. (Our goal is to find a pattern relating our answers to Pascal's Triangle, which is provided above.)

a. Expand $(a+b)^2$.

b. You should have found that $(a+b)^2 = a^2 + 2ab + b^2$. Now expand $(a+b)^3$.

c. You should have found that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Now expand $(a+b)^4$.

d. You should have found that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. What patterns do you notice in the answers to parts **a**–**c** ? Can you find a connection to the combination numbers (Pascal's Triangle numbers)? Explain it.

8. Use the pattern you found in the last question to find each of the following. Suggestion: You already wrote out Pascal's Triangle in a previous problem. Refer to it as you work on these problems.

a. Expand $(a+b)^5$.

b. Expand $(a+b)^7$.

Answers a. $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

b. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

9. The goal of this problem is to expand $(x+2)^4$.

a. Expand $(a+b)^4$.

b. You should have found that $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Now substitute $a=x$ and $b=2$. Then simplify.

Answer $(x+2)^4=x^4+4x^3(2)+6x^2(2)^2+4x(2)^3+(2)^4=x^4+8x^3+24x^2+32x+16$

10. Use the strategy from the last problem to expand each of the following.

a. $(x - 2)^4$

b. $(x + 10)^5$

c. $(x + 10y)^5$