Algebra 2/Pre-Calculus

Name__________________

Permutations and Combinations (Day 2, Pascal's Triangle)

In this problem set, we will explore problems that can be solved using permutations and combinations.

1. A small math class has the following students: Amy, Ben, Catherine, Dan, and Ellen.

a. Suppose the teacher selects a first student as class president, a second student as vice president, and a third student as treasurer. How many ways are there that this can be done?

b. You should have found that there were $5 \cdot 4 \cdot 3 = 60$ ways that this could be done. (5 possible students who could be president, 4 who could be vice president, and 3 who could be treasurer.) Now suppose instead that the teacher is selecting a group of three students to be "class representatives." Are there 60 ways to do this? More than 60? Less than 60? Explain.

c. Notice that in our new problem, when the teacher selects a group of three students, the order doesn't matter. For example, the group of Amy, Ben, and Catherine is the same as the group of Ben, Catherine, and Amy. Therefore, there are less ways to select a group when the order doesn't matter.

How many ways could the group of Amy, Ben, and Catherine be arranged? (In other words, how many orderings are possible for this group?)

d. You should have found that there were 6 ways to arrange the group of Amy, Ben, and Catherine. (ABC, ACB, BAC, BCA, CAB, CBA.) Notice that we have 3 choices for the first student, 2 choices for the second student, and 1 choice for the last student, so our calculation is $3 \cdot 2 \cdot 1 = 6$.

Now consider a group of Catherine, Dan, and Ellen. How many ways can this group be arranged?

e. There are also 6 ways to arrange the 3 students in this group. Would it be accurate to say that any group of 3 students can be arranged in 6 ways? Explain.

f. Let's now revisit the question from part **b**. We had asked how many ways were there for the teacher to select a group of 3 representatives from the 5 students in the class. We said that if the order of the group mattered (the first student is the president, the second student is the vice president, and a third student is the treasurer), then there were $5 \cdot 4 \cdot 3 = 60$ possible ways for the teacher to select the students. But we also said that any group of 3 students could be arranged in $3 \cdot 2 \cdot 1 = 6$ ways.

Now put it all together. How many ways can the teacher select a group of three students from the five students in the class?

g. You should have found that there were 10 ways to do this because $\frac{60}{6}=10$. Here is a visual representation of all of the groups. Notice that there are 60 possible permutations (in which the order mattered) but only 10 distinct groups (in which the order didn't matter.)

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

To summarize, when the order matters, there are $5\cdot 4\cdot 3=60$ different groups. But when the order does not matter, there are only $\frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}=\frac{60}{6}=10$ different groups because there are $3\cdot 2\cdot 1=6$ ways to arrange each group.

2. The idea in the last problem is really important. Take another moment to think about it and make sure it makes sense.

3. There are 8 students in the chess club. Four of these students decide to go to chess camp together. How many ways can this happen?

4. The answer to the last problem is $\frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1680}{24} = 70$. Explain the significance of the numbers 1680 and 24 in the context of this problem.

Answer There are 1680 ways to select a first, a second, a third, and a fourth student from the chess club. In other words, there are 1680 ways to select the four students in order. But the order doesn't matter in this problem. Any group of four students can be put into 24 different orders, so we must divide by 24 to get our answer.

5. There are 25 players on the hockey team. The coach must select 3 of these players as team captains. How many ways can this be done?

Answer $\frac{25 \cdot 24 \cdot 23}{3 \cdot 2 \cdot 1} = \frac{13800}{6} = 2300$

6. A pizzeria offers 12 toppings. How many different pizzas could be made with exactly 2 toppings?

Answer $\frac{12 \cdot 11}{2 \cdot 1} = 66$

Combinations and Permutations

The solutions to the last group of problems are all calculated in the same way. We call this calculation a ***combination***. The notation for combinations is similar to permutations, as shown in the following examples:

$${}_{10}C_3 = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} \qquad {}_{20}C_6 = \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \qquad {}_{8}C_1 = \frac{8}{1}$$

There is also alternative notation for combinations that is frequently used, as shown in the examples below:

$$\binom{10}{3} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} \qquad \binom{20}{6} = \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \qquad \binom{8}{1} = \frac{8}{1}$$

One more observation: When deciding whether to use a permutation or a combination to solve a problem, ask yourself whether the order matters. (Order matters in permutations but not in combinations.)

7. Find each of the following:

a. ${}_{7}C_{5}$

b. $\binom{700}{2}$

c. ${}_{4}C_{4}$

Answers a. 21 b. 244,650 c. 1

8. The calculator has a special button for combinations. To evaluate ${}_{7}C_{5}$ on your calculator, you will need to press the MATH button. Then arrow over to the PRB menu and select the second option, ${}_{n}C_{r}$. Depending on which calculator you have, this may appear on your screen as 7nCr5 or ${}_{7}C_{5}$. Either way, press ENTER and the answer should appear.

Answer 21

9. The calculator also has a special button for permutations. Use the calculator to find ${}_7P_5$.
Note: The steps are almost exactly the same as the steps for finding ${}_7C_5$.

Answer 2520

10. Each of the following problems can be solved with exponents, factorials, permutations, or combinations. ***Note:*** Answers are provided at the end of this problem.

a. A pizzeria offers 10 toppings. How many three topping pizzas are possible?

b. A committee of 25 people must choose a 5 person subcommittee. How many ways can this be done?

c. A baseball team has 13 players. The coach must select 9 of these players for a batting order. (One player bats first, another bats second, etc.) How many batting orders are possible?

d. An ice cream shop serves 12 flavors. How many three scoop dishes are possible if no flavors are repeated?

e. An ice cream shop serves 12 flavors. How many three scoop ice cream cones are possible if no flavors are repeated?

f. Briefly explain why we used combinations when the ice cream was served in dishes whereas we used permutations when the ice cream was served as a cone.

g. A coin is flipped 10 times. How many sequences of heads and tails are possible?

h. A coin is flipped 10 times. How many sequences of flips include exactly 4 heads?

i. If a coin is flipped 10 times, what is the probability that it will land heads exactly 4 times?

Answers a. ${}_{10}C_3 = \frac{10\cdot9\cdot8}{3\cdot2\cdot1} = 120$ b. ${}_{25}C_5 = \frac{25\cdot24\cdot23\cdot22\cdot21}{5\cdot4\cdot3\cdot2\cdot1} = 53130$

c. ${}_{13}P_9 = 13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5 = 259{,}459{,}200$ d. ${}_{12}C_3 = \frac{12\cdot11\cdot10}{3\cdot2\cdot1} = 220$

e. ${}_{12}P_3 = 12\cdot11\cdot10 = 1320$ f. The order of the flavors matters in an ice cream cone, but it doesn't matter in an ice cream dish. Eating an ice cream cone with chocolate on top, then vanilla, and then strawberry is a different experience from eating an ice cream cone that has strawberry on top, then vanilla, then chocolate. But when those flavors are served in a dish, they don't have an order. g. $2^{10} = 1024$

h. ${}_{10}C_4 = \frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1} = 210$ i. $\frac{210}{1024}$

11. Determine whether each of the following statements is true or false. If the statement turns out to be false, see if you can fix to make it correct. ***Hint:*** If you're not sure, try some numbers.

a. $_nC_r = \frac{_nP_r}{r!}$

b. $_nP_r = \frac{n!}{(n-r)!}$

c. $_nC_r = \frac{n!}{r!(n-r)!}$

Answers a. True b. True c. True

12. Optional Challenge Prove each of the statements from problem **11**.

13. Consider the following recursively defined function:

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 0 \\ n \cdot f(n-1) & \text{if} \quad n > 0 \end{cases}$$

a. Find $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

b. What is the more common name for this function?

c. The recursive function about is one way of defining $n!$ What is $0!$ according to this definition?

Answers a. $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, $f(3) = 6$, $f(4) = 24$ b. $f(n) = n!$
c. $0! = 1$ (This is usually how $0!$ is defined. The reasons for this are beyond the scope of this course.)

14. Which is bigger: ${}_nC_r$ or ${}_nP_r$? Is this always true? Explain.

Answer In general, ${}_nP_r$ is bigger than ${}_nC_r$, but they can be equal. For example, ${}_1C_1 = {}_1P_1 = 1$.

15. Suppose *x*, *y*, and *z* are integers and $x < y < z$.

a. Which is bigger: ${}_zP_x$ or ${}_zP_y$? Explain.

Answer ${}_zP_y > {}_zP_x$. This is always true. Can you explain why???

b. Which is bigger: ${}_zC_x$ or ${}_zC_y$? Explain.

Answer We can't tell which of these is bigger. Can you explain why???