Algebra 2/Pre-Calculus

Name__________________

Introduction to Pascal's Triangle (Day 3, Pascal's Triangle)

Pascal's Triangle

The goal of this handout is to explore the properties of Pascal's Triangle.

Consider the following pattern:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1. What is the pattern? Explain in words.

2. Add the next four rows to the triangle.

Connection to the Combination Numbers

In our last problem set, we introduced the concepts of factorials, permutations, and combinations. The definitions for each of these are given below.

Factorials: $$n! = \begin{cases} 1 & \text{if} \quad n = 0 \\ n\cdot(n-1)\cdots 3\cdot 2\cdot 1 & \text{if} \quad n > 0 \end{cases}$$

Permutations: $${}_nP_r = \frac{n!}{(n-r)!}$$

Combinations: $$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}$$

Notice that there are two notations for combination numbers: $\binom{n}{r}$ and ${}_nC_r$. These both mean the same thing and can be used interchangeably.

There is an important connection between Pascal's Triangle and the combination numbers, as we will see in the next problem.

3. Find each of the following. ***Note:*** Each question is asking you for multiple values. Can you find the pattern?

a. Find $\binom{0}{0}$. ***Remember:*** $0! = 1$

b. Find $\binom{1}{0}$ and $\binom{1}{1}$

c. Find $\binom{2}{0}$, $\binom{2}{1}$, and $\binom{2}{2}$.

d. Find $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.

e. Find $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$

f. How do these answers relate to Pascal's Triangle?

g. Use Pascal's triangle to find $\binom{5}{0}$, $\binom{5}{1}$, $\binom{5}{2}$, $\binom{5}{3}$, $\binom{5}{4}$, and $\binom{5}{5}$.

h. **Optional Challenge** Can you explain why this is happening?

Properties

Pascal's Triangle is sometime written in the following way.

$$\begin{array}{c} {}_0C_0 \\ {}_1C_0 \;\; {}_1C_1 \\ {}_2C_0 \;\; {}_2C_1 \;\; {}_2C_2 \\ {}_3C_0 \;\; {}_3C_1 \;\; {}_3C_2 \;\; {}_3C_3 \\ {}_4C_0 \;\; {}_4C_1 \;\; {}_4C_2 \;\; {}_4C_3 \;\; {}_4C_4 \\ {}_5C_0 \;\; {}_5C_1 \;\; {}_5C_2 \;\; {}_5C_3 \;\; {}_5C_4 \;\; {}_5C_5 \\ {}_6C_0 \;\; {}_6C_1 \;\; {}_6C_2 \;\; {}_6C_3 \;\; {}_6C_4 \;\; {}_6C_5 \;\; {}_6C_6 \end{array} \quad = \quad \begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

4. a. Find the value of $\binom{5}{1} + \binom{5}{2}$.

b. Find the value of $\binom{6}{2}$.

c. You should have found that you got the same answer for parts **a** and **b**. Why did this happen? ***Hint:*** Think about where each of these three numbers is on Pascal's Triangle.

d. Which of the following statements is true:

$$\binom{9}{6}+\binom{9}{7}=\binom{10}{6} \text{ or } \binom{9}{6}+\binom{9}{7}=\binom{10}{7}?$$

e. Which of the following statements is true:

$$\binom{n-1}{r-1}+\binom{n-1}{r}=\binom{n}{r-1} \text{ or } \binom{n-1}{r-1}+\binom{n-1}{r}=\binom{n}{r}?$$

f. Fill in the question marks to make a true statement: $\binom{n}{r}+\binom{n}{r+1}=\binom{?}{?}$

Answers a. $\binom{5}{1}+\binom{5}{2}=5+10=15$ b. $\binom{6}{2}=15$ c. Look at rows 5 and 6 of Pascal's Triangle. 5 and 10 are above 15. d. $\binom{9}{6}+\binom{9}{7}=\binom{10}{7}$

e. $\binom{n-1}{r-1}+\binom{n-1}{r}=\binom{n}{r}$ f. $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$

5. **Optional Challenge** In the last problem, we made the following observation:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Our goal is to prove this. Remember, $\binom{n}{r}$ is defined to be $\frac{n!}{r!(n-r)!}$.

a. Explain why $(n+1)! = (n+1) \cdot n!$ (You will use this idea later in this proof.)

b. Explain why $n(n+1)! = n(n!) + n!$ (You will use this idea later in this proof.)

c. Show that $\binom{n}{r+1} = \frac{n!}{(r+1)!(n-r-1)!}$

d. Show that $\binom{n+1}{r+1} = \frac{(n+1)!}{(r+1)!(n-r)!}$.

e. Now we must add $\frac{n!}{r!(n-r)!}+\frac{n!}{(r+1)!(n-r-1)!}$. What will the common denominator be?

f. Show that $\frac{n!}{r!(n-r)!}+\frac{n!}{(r+1)!(n-r-1)!}=\frac{r(n!)+(n!)+n(n!)-r(n!)}{(r+1)!(n-r)!}$

g. Finish proving that $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$