

## Algebra 2/Pre-Calculus

Name \_\_\_\_\_

### Factorials and Permutations (Day 1, Pascal's Triangle)

In this handout, we will review the principles of factorials, permutations, and combinations. (You probably learned about these concepts in a previous math class, but we will begin this unit with a review of those ideas.) In particular, we will explore how these concepts can be used to solve problems involving counting and probability.

### The Multiplication Principle

1. In order to raise money for the prom, the senior class is selling T-shirts. The T-shirts come in three colors (white, yellow, or blue) and four sizes (small, medium, large, and extra large). How many type of T-shirts are there?

**Answer**  $3 \cdot 4 = 12$

2. Ben was buying a turkey sandwich at a local deli. The sandwich came with a choice of cheese (Swiss, American, cheddar, or provolone), a choice of side (fries, baked potato, soup, or salad) and a choice of dessert (brownie or cookie). How many possible variations are there for Ben's meal?

**Answer**  $4 \cdot 4 \cdot 2 = 32$

3. Mr. Verner owns 2 pairs of shoes, 5 pairs of pants, 11 shirts, and 6 sweaters. How many "outfits" can Mr. Verner wear to school?

**Answer**  $2 \cdot 5 \cdot 11 \cdot 6 = 660$

4. Was the last problem completely realistic? Did we have to make any assumptions in order to answer the question? Explain.

**Answer** In the last problem, we had to make a lot of simplifying assumptions. For example, we assumed that Mr. Verner was going to wear a sweater everyday, which is not necessarily the case. We also assumed that Mr. Verner would be willing to wear any of his pants with any of his shirts, but he might be avoid certain pairings that won't look good together. (Mr. Verner's has impeccable style.) In mathematics, we frequently need to make simplifying assumptions in order to solve problems. This is fine, but we should be aware of the assumptions that we are making.

5. Problems 1 – 3 of this handout illustrate an idea that is frequently called "the multiplication principle." Explain what is meant by the phrase "the multiplication principle" in the context of these types of problems.

## Use of Exponents

6. Suppose you flipped a coin 3 times.

- a. How many sequences of heads and tails are possible? (We consider HHT different from HTH because the heads and tails are in a different order.) *Hint:* How many possibilities are there for the first flip? For the second? For the third?

- b. Hopefully, you found the answer to part a was 8. (2 possibilities for each flip, so  $2 \cdot 2 \cdot 2 = 2^3 = 8$ .) Here's a way of writing them:

HHH	HHT	HTT	TTT
	HTH	THT	
	THH	TTH	

Now suppose the coin is flipped 4 times. How many sequences of heads and tails are possible? (You do not have to list them.)

- c. You should have found that there were 16 sequences. ( $2^4 = 16$ .) Suppose a coin is flipped 5 times. What is the probability that it will land tails every time?

**Answers** a. 8 b. 16 c. There are 32 possibilities, so the probability of all tails is  $\frac{1}{32}$ .

7. A website requires a three digit code. Here are some examples of possible codes: 123, 050, 777, 098, etc.
- a. How many different codes are possible? *Hint:* How many possibilities are there for the first digit? The second? The third?
- b. A different website requires a code consisting of 4 letters (A, B, C, D). Here are some examples of possible codes: ABCD, DCBA, AADD, CCCC. How many codes are possible?

**Answers** a.  $10 \cdot 10 \cdot 10 = 10^3 = 1000$    b.  $4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = 256$

In the last two problems, we saw how exponents could be used to answer certain types of counting questions. In the next problem, we will combine the use of exponents with the idea of the multiplication principle.

8. Another website requires a code consisting of 4 letters (A, B, C, D), followed by three numbers (1, 2, 3). Here are some examples of possible codes: ABCD123, AABB332, DDDD111. How many codes are possible?

**Answer**  $4 \cdot 4 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 4^4 \cdot 3^3 = 6,912$

9. This problem deals with different license plates.
- A certain type of Massachusetts license plate follows the pattern of four digits, followed by two letters.<sup>1</sup> Here are some license plates of this type: 5540AB, 0987ZY, 0000QQ. How many license plates of this type are possible?
  - A different type of Massachusetts license plate follows the pattern of two digits, then two letters, then two digits. Here are some license plates of this type: 45AN38, 00BB77, 12XY03. How many license plates of this type are possible?
  - Many California license plates follow the pattern of a digit, then three letters, then three numbers. How many license plates of this type are possible?

**Answers** a.  $10 \cdot 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 = 10^4 \cdot 26^2 = 6,760,000$

b.  $10 \cdot 10 \cdot 26 \cdot 26 \cdot 10 \cdot 10 = 10^4 \cdot 26^2 = 6,760,000$     c.  $10^4 \cdot 26^3 = 175,760,000$

10. This problem involves a variation on the technique we used in the last few problems.
- A certain website requires a five character code. The characters must all be numbers or they must all be letters. (So 12005 and XYZAA would be valid codes, but 12AB3 would not be a valid code.) How many codes are possible?
  - A different website requires a code that is either a two letters followed by three digits or a single letter followed by four digits. (For example, AB122 and C0987 would be valid codes, but ABC12 would not be a valid code.) How many codes are possible?

**Answers** a.  $10^5 + 26^5 = 11,981,376$     b.  $26^2 \cdot 10^3 + 26 \cdot 10^4 = 936,000$

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<sup>1</sup> In actuality, the rules for license plates are fairly complicated. We have simplified these rules considerably to make this problem more accessible.

## Factorials and Permutations

11. Ben, Jermaine, and Mudit each had to give a presentation in math class. How many different orders are there in which they could give their presentations? List them. **Note:** Make sure your list is organized in a clear way. Don't just list people randomly.

**Answer** 6 possibilities. Listed in alphabetical order: BJM, BMJ, JBM, JMB, MBJ, MJB

12. Fiona, Giulia, Kathryn, and Mei also had to give presentations in math class. How many different orders are there in which they could give their presentations? (You do not need to list them.)

**Answer** 24 possibilities.

13. We can find the answer to problem 11 by performing the calculation  $3 \cdot 2 \cdot 1 = 6$  and we can find the answer to problem 12 by performing the calculation  $4 \cdot 3 \cdot 2 \cdot 1 = 24$ . Explain why this approach works. Make sure your explanation is clear!

14. A website requires six letter code. Each of the letters A, B, C, D, E, F must be used exactly once. How many codes are possible?

**Answer**  $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

15. To express calculations like the one in the last problem more concisely, we use the **factorial** symbol. This symbol is an exclamation point and it is used in the following way.

$$3! = 3 \cdot 2 \cdot 1 = 6 \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \quad 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Find the values of  $9!$  and  $13!$  by using the calculator. **Note:** To find  $9!$  start enter a 9 on the standard screen. Then press the MATH button and arrow over the PRB menu. Choose the fourth option. Your screen should now say "9!" press ENTER and the answer will appear. You can find  $13!$  in a similar manner.

**Answers**  $9! = 362,880$  and  $13! = 6,227,020,800$

16. Another website requires a four letter code. Any letter may be used, but no letter can be used more than once. How many codes are possible? **Hint:** How many possibilities are there for the first letter? Once the first letter is chosen, how many possibilities are there for the second letter? The third? The fourth?

**Answer**  $26 \cdot 25 \cdot 24 \cdot 23 = 358,800$

17. In the last problem, there were 26 possible choices for the first letter. Once that letter was used, there were 25 possibilities for the next letter, 24 possibilities for the third letter, and 23 possibilities for the fourth letter. Thus, the total number of possibilities can be calculated by multiplying  $26 \cdot 25 \cdot 24 \cdot 23 = 358,800$ .

Suppose a different code is requires 7 digits, none of which can be repeated. How many different codes are possible?

**Answer**  $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 604,800$

18. Mr. Verner selects one student from his class to put the solution for problem 1 on the board, another student to put problem 2 on the board, and a third to put problem 3 on the board. There are 25 students in his class and no one is required to put more than one problem on the board. How many ways can this happen?

**Answer**  $25 \cdot 24 \cdot 23 = 13,800$

19. The calculation we were performing in the last few problems is called a **permutation**. We use a special notation for permutations. In the last problem, instead of writing  $25 \cdot 24 \cdot 23$ , we simply write  ${}_{25}P_3$ .

Find the value for each of the following permutations.

a.  ${}_{20}P_6$

b.  ${}_{50}P_2$

c.  ${}_4P_4$

d.  ${}_{100}P_1$

**Answers** a.  ${}_{20}P_6 = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 = 27,907,200$  b.  ${}_{50}P_2 = 50 \cdot 49 = 2,450$

c.  ${}_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$  d.  ${}_{100}P_1 = 100$

20. Is  ${}_8P_3$  equal to  $\frac{8!}{5!}$ ? Explain how you can tell without using your calculator.

**Answer**  $\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = {}_8P_3$

21. Use the approach from the last problem to show how you could rewrite  ${}_{10}P_6$  using the factorial symbol. **Show how you got your answer!**

**Answer**  ${}_{10}P_6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{4!}$

21. Is  ${}_nP_r$  equal to  $n(n-1)(n-2)\cdots(n-r)$ ? Explain. **Hint:** Think about an easy example like  ${}_{10}P_3$ .

**Answer**  ${}_nP_r \neq n(n-1)(n-2)\cdots(n-r)$ . Notice that  ${}_{10}P_3 = 10 \cdot 9 \cdot 8 \neq 10(10-1)(10-2)(10-3)$

22. How can you fix the statement in the last problem?

**Answer**  ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$ .

23. Show how you could rewrite  ${}_nP_r$  using the factorial symbol. **Hint:** Think about how you rewrote  ${}_{10}P_6$  back in problem 21.

**Answer**  ${}_nP_r = \frac{n!}{(n-r)!}$  (Just like  ${}_{10}P_6 = \frac{10!}{4!}$  back in problem 21.)

24. Determine whether each of the following statements is true or false. If the statement turns out to be false, see if you can fix to make it correct. **Hint:** If you're not sure, try some numbers.

a.  $n! = n(n-1)(n-2)\cdots(3)(2)(1)$

b.  ${}_n P_r = n(n-1)(n-2)\cdots(n-r)$

c.  ${}_n P_r = \frac{n!}{(n-r)!}$

d.  ${}_n P_1 = n$

e.  ${}_n P_n = 1$

f.  ${}_n P_n = n!$

**Answers** a. True b. False. It should be  ${}_n P_r = n(n-1)(n-2)\cdots(n-r+1)$  c. True d. True  
e. False f. True

25. More practice! Find each of the following.

- a. 20 students participated in an essay contest. The top three entries are awarded first place, second place, or third place. How many ways can this happen?
- b. A small class has 6 students, all of whom must give presentations. How many orders are possible for the student presentations?
- c. A coin is flipped 6 times. What is the probability that it lands heads every time?
- d. An ice cream store offers 7 flavors: Vanilla (V), Chocolate (C), Mint (M), Hazelnut (H), Strawberry (S), Raspberry (R), and Oreo (O). How many three scoop ice cream cones are possible? **Note:** Each of the following is considered a different cone: VCM, MCV, SRS, SSR, HHH.
- e. How many ice cream cones are possible if no flavors are repeated? (VCM and MCV are still allowed, but SRS, SSR, and HHH are no longer permitted.)
- f. A band knows 12 songs. They must decide what order they will play these songs in at their concert. How many different orders are possible? (Assume that each song gets played exactly once.)

**Answers** a.  ${}_{20}P_3 = 20 \cdot 19 \cdot 18 = 6840$    b.  $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$    c.  $\frac{1}{2^6} = \frac{1}{64}$

d.  $7^3 = 343$    e.  ${}_7P_3 = 7 \cdot 6 \cdot 5 = 210$    f.  $12! = 479,001,600$

**26. Optional Challenge** These problems look ahead to an idea that we will explore in our next handout. Try them out! (But don't worry if you aren't sure how to do them.)

- a. A small math class has five students: Amy, Ben, Catherine, Dan, and Ellen. The teacher decides to select a group of three of these students as class representatives. How many ways can the teacher select the groups? **Note:** Try to do this problem **without** using your calculator.

- b. A larger math class has 10 students. The teacher decides to select a group of 4 of these students as class representatives. How many ways can this be done? **Note:** Try to do this problem **without** using any of the special features on the calculator. (Only use addition, subtraction, multiplication, and/or division.)

**Answers** a. 10 b. 210