In this handout, we will observe patterns for factoring the sum or difference of cubes.

1. Perform the following divisions. Use long division.

   a. \[ \frac{x^3 + 5x^2 + 3x - 2}{x + 3} \]
   b. \[ \frac{x^3 - 14x - 19}{x - 4} \]
   c. \[ \frac{x^4 - x^3 - 12x^2 - x + 10}{x^2 + 2x - 2} \]
   d. \[ \frac{x^4 - x^3 - 2x^2 + 3x - 3}{x^2 - x + 1} \]

Answers:

a. \[ x^2 + 2x - 3 + \frac{1}{x+3} \]

b. \[ x^2 + 4x + 2 - \frac{11}{x-4} \]

c. \[ x^2 - 3x - 4 + \frac{x+2}{x^2+2x-2} \]

d. \[ x^2 - 3 \]
2. The goal of this problem is to factor $x^3 - 8$. Note: Answers are provided at the end of this problem.

a. Perform the following division: $\frac{x^3 - 8}{x - 2}$. \textit{Hint:} This is the same as $\frac{x^3 + 0x^2 + 0x - 8}{x - 2}$.

b. Now factor $x^3 - 8$.

\textbf{Answers} a. $x^2 + 2x + 4$ \quad b. $(x - 2)(x^2 + 2x + 4)$

3. The goal of this problem is to factor $x^3 - 27$. Note: Answers are provided at the end of this problem.

a. Perform the following division: $\frac{x^3 - 27}{x - 3}$. \textit{Hint:} Same hint as the last problem.

b. Now factor $x^3 - 27$.

\textbf{Answers} a. $x^2 + 3x + 9$ \quad b. $(x - 3)(x^2 + 3x + 9)$
4. The goal of this problem is to factor \( x^3 - 1000 \). \textit{Note:} Answers are provided at the end of this problem.

a. Here’s a summary of what you found in the last two problems:
\[
x^3 - 8 = (x - 2)(x^2 + 2x + 4)
\]
\[
x^3 - 27 = (x - 3)(x^2 + 3x + 9)
\]

See if you can identify a pattern and use it to factor \( x^3 - 1000 \). \textit{Note:} If you need a hint, look ahead to part b.

b. Perform the following division: \( \frac{x^3 - 1000}{x - 10} \). \textit{Hint:} Same hint as the last problem.

c. Now factor \( x^3 - 1000 \). (If you haven’t already.)

\textbf{Answers} a. \((x - 10)(x^2 + 10x + 100)\) b. \(x^2 + 10x + 100\) c. \((x - 10)(x^2 + 10x + 100)\)
5. Here’s a summary of what you found in the last three problems:
   \[ x^3 - 8 = (x - 2)(x^2 + 2x + 4) \]
   \[ x^3 - 27 = (x - 3)(x^2 + 3x + 9) \]
   \[ x^3 - 1000 = (x - 10)(x^2 + 10x + 100) \]
   You should be seeing a pattern. Use it to factor each of the following. Note: Answers are provided at the end of this problem.
   a. Factor \( x^3 - 1 \)
   
   b. Factor \( x^3 - 64 \)

   Answers a. \((x - 1)(x^2 + x + 1)\)  b. \((x - 4)(x^2 + 4x + 16)\)

6. The goal of this problem is to find a rule for the difference of cubes.
   a. Factor \( a^3 - b^3 \). Note: Look ahead to part b if you need a hint.
   
   b. Multiply: \((a - b)(a^2 + ab + b^2)\)

   c. You should have found that \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \). This gives us a rule for factoring the difference of cubes. Use this rule to factor \( x^3 - 125 \). Hint: What’s \( a \)? What’s \( b \)?
7. The goal of this problem is to find a factoring for $x^3 + 8$. \textbf{Note:} Answers are provided at the end of this problem.

   a. Perform the following division: \( \frac{x^3 + 8}{x + 2} \). \textbf{Reminder:} Write it as $x^3 + 0x^2 + 0x + 8$.

   b. Factor $x^3 + 8$.

\textbf{Answers} a. $x^2 - 2x + 4$  b. $(x + 2)(x^2 - 2x + 4)$

8. The goal of this problem is to find a factoring for $x^3 + 27$. \textbf{Note:} Answers are provided at the end of this problem.

   a. Perform the following division: \( \frac{x^3 + 27}{x + 3} \).

   b. Factor $x^3 + 27$.

\textbf{Answers} a. $x^2 - 3x + 9$  b. $(x + 3)(x^2 - 3x + 9)$
9. The goal of this problem is to factor $x^3 + 1000$. \textbf{Note:} Answers are provided at the end of this problem.

a. Here’s a summary of what you found in the last two problems:

\[
x^3 + 8 = (x + 2)(x^2 - 2x + 4)
\]
\[
x^3 + 27 = (x + 3)(x^2 - 3x + 9)
\]

See if you can identify a pattern and use it to factor $x^3 + 1000$. \textbf{Note:} If you need a hint, look ahead to part b.

b. Perform the following division: $\frac{x^3 + 1000}{x + 10}$. \textbf{Hint:} Same hint as the last problem.

c. Now factor $x^3 + 1000$. (If you haven’t already.)

\textbf{Answers} a. $(x + 10)(x^2 - 10x + 100)$ b. $x^2 - 10x + 100$ c. $(x + 10)(x^2 - 10x + 100)$
10. Here’s a summary of what you found in the last three problems:

\[ x^3 + 8 = (x + 2)(x^2 - 2x + 4) \]
\[ x^3 + 27 = (x + 3)(x^2 - 3x + 9) \]
\[ x^3 + 1000 = (x + 10)(x^2 - 10x + 100) \]

You should be seeing a pattern. Use it to factor each of the following. Note: Answers are provided at the end of this problem.

a. Factor \( x^3 + 125 \)

b. Factor \( x^3 + 1 \)

Answers a. \((x + 5)(x^2 - 5x + 25)\) b. \((x + 1)(x^2 - x + 1)\)

11. The goal of this problem is to find a rule for the sum of two cubes.

a. Factor \( a^3 + b^3 \). Note: Try to find a rule that’s similar to the rule for the difference of cubes. Look ahead to part b if you need a hint.

b. Multiply: \((a + b)(a^2 - ab + b^2)\)

c. You should have found that \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \). This gives us a rule for factoring the sum of two cubes. Use this rule to factor \( x^3 + 64 \) and \( x^3 + 8000 \). Hint: What’s \( a \)? What’s \( b \)?
12. Completely factor each of the following. **Note:** Answers are provided at the end of this problem.

   a. $3x^4 - 24x$
   b. $x^3 - 5x^2 + 6x - 30$

   c. $3x^2 - 8x + 4$
   d. $-x^3 - 27$

   e. $5x^3 + x^2 - 6x$
   f. $9x^2 - 16$

   g. $4x^5 - 4x^4 - 3x^3$
   h. $(x^2 - 2x - 35)(x^2 - 49)$

i. **Optional Challenge:** $x^6 - 64$

**Answer**

a. $3x(x - 2)(x^2 + 2x + 4)$

b. $(x^2 + 6)(x - 5)$

c. $(3x - 2)(x - 2)$

d. $-(x + 3)(x^2 - 3x + 9)$

e. $x(5x + 6)(x - 1)$

f. $(3x + 4)(3x - 4)$

g. $x^3(2x + 1)(2x - 3)$

h. $(x + 5)(x + 7)(x - 7)^2$

i. $(x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4)$
13. **Optional Challenge Problem** The goal of this problem is to generalize the factoring patterns you explored in the previous problems to polynomials with bigger exponents.

*Note:* You can check your answers for each of these problems by typing it into wolframalpha and scrolling down to “alternate forms.” *Another note:* You will probably need to do the work these problems on another sheet of paper.

a. Factor $8x^3 - 27$.

b. Factor $(x + 5)^3 - 8$.

c. Factor $(x + 3)^2 - 7(x + 3) + 12$.

d. Factor $x^5 - 32$. **Hint:** $\frac{x^5 - 32}{x - 2}$.

e. Factor $x^5 - 100000$

f. Factor $x^5 + 32$

g. Factor $x^5 - 1$

h. Factor $x^9 + 1$